Diffraction of surface waves on an incompressible fluid

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An explicit determination is made of the velocity potential for the small-amplitude time-periodic excitation of a bottomless, heavy, incompressible fluid which results from an internal point source, assuming that the equilibrium free surface lies in a horizontal plane and taking account of the presence within the fluid of a thin rigid plane with a vertical straight edge. The surface-wave component of the potential is expressed by means of single quadratures for any relative disposition of the source and observation points and, apart from exponential factors involving the depths of the respective points, proves to have the same functional character as the steady-state velocity potential for the acoustic (or compressional) motion which is sustained by an infinite line source parallel to the straight edge of a thin rigid half-plane.

1. Introduction

Waves of irrotational type can be propagated along the free surface of an ideal incompressible fluid in a gravitational field, and a theory fashioned with space harmonic functions holds the key to both the details of their form and excitation. If the linearized approximation to the free surface boundary condition is invoked, the concomitant theory of small amplitude wave motion can be developed in a wide variety of (two- and three-dimensional) circumstances. Recently, for instance, Voit (1961) has undertaken to consider the small amplitude waves generated by a periodically pulsating point source within a fluid of infinite depth, and to calculate their diffraction at a thin rigid plane having a vertical straight edge. The configuration is indicated schematically in figure 1, where the half-space $z < 0$ is occupied by the fluid (whose equilibrium free surface lies in the plane $z = 0$) and where the z-axis is coincident with the edge of the semi-infinite plane $x > 0$, $y = 0$; the source of excitation in the fluid is located at the point Q, whose cylindrical co-ordinates are $(r', \theta', -h)$.

When the state of fluid motion is expressed by a velocity potential which is the real part of $\phi(\mathbf{r})\, e^{-i\omega t}$, the spatial factor $\phi(\mathbf{r})$ satisfies Laplace's equation

$$\nabla^2\phi = 0, \tag{1}$$

and the linearized free-surface approximation takes the form

$$\partial\phi/\partial z = K\phi, \quad z = 0 \quad \text{where} \quad K = \omega^2/g, \tag{2}$$

while the condition of vanishing normal velocity at either side of the rigid half-plane requires that

$$\partial\phi/\partial\theta = 0, \quad \theta = 0 \quad \text{and} \quad 2\pi. \tag{3}$$

In addition to the foregoing differential conditions, the function ϕ itself has the prescribed behaviour:

$$\phi \to 1/4\pi R, \quad R^2 \equiv R_{PQ}^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z+h)^2 \tag{4}$$

near the source point Q, and

$$\phi \to 0, \quad z \to -\infty \tag{5}$$

far below the surface.

Starting with the integral representation of the singular part of the potential,

$$\frac{1}{4\pi R} = \frac{1}{4\pi}\int_0^\infty e^{-k(z+h)} J_0(k\{r^2 + r'^2 - 2rr' \cos(\theta - \theta')\}^{\frac{1}{2}})\, dk \quad (z > -h), \tag{6}$$

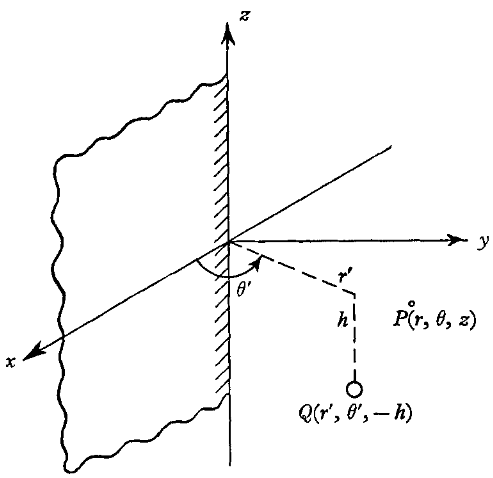

FIGURE 1. The diffracting half-plane and co-ordinate system.

where J_0 denotes the zero-order Bessel function, Voit establishes the form

$$\phi(r,\theta,z) = \frac{1}{4\pi}\left[\int_0^\infty e^{-k(z+h)}\,\psi(r,\theta;\,k)\,dk + \int_0^\infty \frac{k+K}{k-K} e^{-k(h-z)}\,\psi(r,\theta;\,k)\,dk\right] \quad (z > -h), \tag{7}$$

which fulfils the free-surface condition (2), the asymptotic behaviour (5) being left as a condition on the potential representation in the region $z < -h$. The harmonic nature of ϕ implies that the function $\psi(r,\theta;\,k)$ obeys a Helmholtz equation

$$(\nabla_{r,\theta}^2 + k^2)\,\psi = 0, \tag{8}$$

wherein k plays a parametric role, and Voit follows Sretenskii (1959) in calling on the Sommerfeld technique (of long standing in acoustical and optical diffraction theory) to obtain the two-branched solution of (8) which has a vanishing normal derivative at $\theta = 0$ and 2π, namely

$$\psi(r,\theta;\,k) = \begin{cases} J_0(kD) + J_0(k\bar{D}) + V(r,\theta;\,k), & 0 < \theta < \pi - \theta', \\ J_0(kD) + V(r,\theta;\,k), & \pi - \theta' < \theta < \pi + \theta', \\ V(r,\theta;\,k), & \pi + \theta' < \theta < 2\pi, \end{cases} \tag{9}$$

where

$$V(r,\theta;\,k) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{e^{\frac{1}{2}i(\theta'-\theta)+\frac{1}{2}\eta}+e^{-\frac{1}{2}i(\theta'-\theta)-\frac{1}{2}\eta}}\right.$$

$$\left.+\frac{1}{e^{\frac{1}{2}i(\theta'+\theta)-\frac{1}{2}\eta}+e^{-\frac{1}{2}i(\theta'-\theta)+\frac{1}{2}\eta}}\right]J_0(k\tilde{D})\,d\eta, \qquad (10)$$

and

$$D^2 = r^2+r'^2-2rr'\cos(\theta-\theta'), \quad \bar{D}^2 = r^2+r'^2-2rr'\cos(\theta+\theta'),$$

$$\tilde{D}^2 = r^2+r'^2+2rr'\cosh\eta.$$

Once the velocity potential is known, the displacement of the free surface, ζ, is obtainable from the expression

$$\zeta = -(i\omega/g)\,\phi(r,\theta,0)\,e^{-i\omega t}, \qquad (11)$$

or, after recourse to (7), from

$$\zeta = -\frac{i\omega}{2\pi g}e^{-i\omega t}\int_0^{\infty}\frac{e^{-kh}}{k-K}\psi(r,\theta;\,k)\,k\,dk. \qquad (12)$$

Relying on the latter form, in conjunction with the representation (9), Voit singles out the 'required diffraction' contribution, viz:

$$\zeta_{\text{diff.}} = -\frac{i\omega}{2\pi g}e^{-i\omega t}\int_0^{\infty}\frac{e^{-kh}}{k-K}V(r,\theta;\,k)\,k\,dk, \qquad (13)$$

and proceeds to an asymptotic estimate of the double integral stemming therefrom (by substitution of (10)), with the assumptions that $\omega^2 d/g \gg 1$, where d is a lower bound to r, r' and R, and that both source and observation points are sufficiently far from the edge of the half-plane. The outcome of this elaborate analysis is contained in the result

$$\zeta_{\text{diff.}} \sim \frac{\omega}{2\pi^{\frac{3}{2}}g}\frac{\exp(-\omega^2 h/g)}{(rr')^{\frac{1}{2}}}\exp\left[-i\omega\{t-\omega(r+r')/g\}\right]\frac{\cos\frac{1}{2}\theta\cos\frac{1}{2}\theta'}{\cos\frac{1}{2}(\theta-\theta')\cos\frac{1}{2}(\theta+\theta')}, \qquad (14)$$

applicable for angles θ which are not close to $\pi-\theta'$ or $\pi+\theta'$, these defining the directions of specular reflexion from, and grazing incidence on the edge of, the half-plane.

It is the purpose of this communication to demonstrate that the surface-wave component of the fluid motion in the circumstances envisaged above can be given in a simpler and more perspicuous form which facilitates its calculation for any relative disposition of the source and observation points. Apart from a multiplicative constant and an exponential factor characteristic of surface-wave amplitude variation along a direction normal to the guiding surface, the aforesaid component proves to be formally identical with the acoustic velocity potential arising from an infinitely extended line source parallel to the edge of a rigid half-plane;† as befits the propagation of surface waves over an incompressible fluid,

† Penney & Price (1952) observed that the two-dimensional acoustical or optical solutions for *plane* wave diffraction at a half-plane could be taken over for the purpose of studying the diffraction of time-periodic straight-crested surface waves at vertical breakwaters of the rigid or cushion types.

the magnitude of the wave vector which figures in this equivalent two-dimensional wave complex is $K = \omega^2/g$. The line source (or Green's) function for the half-plane configuration is available in terms of compact single integrals (which were originally deduced by Macdonald 1915), and its behaviour at various points of space—including the regions of transition according to geometrical propagation theory—is a matter of record.

In this paper a direct method is employed to determine the complete Green's function for the hydrodynamical source problem, relying on considerations of spatial symmetry, Fourier expansion and Fourier–Bessel transformation. The practical advantage of enumerating configurations which possess an even or odd symmetry in relation to the plane domain $x < 0$, $y = 0$, while adhering to the stipulated (normal derivative) boundary condition on $x > 0$, $y = 0$, is that attention can be focused initially on the region $y > 0$ (or $0 < \theta < \pi$) to one side of of the diffracting half-plane. Included in the source functions for each of the symmetry types are contributions of a surface wave and a non-propagating (potential) nature; the former are easily isolated and only their average is needed to evolve the result described above. The details of the fundamental even–odd symmetry functions are set forth in the next sections and then the solution of the given problem is effected forthwith.

2. Source-function of even symmetry

The function envisaged under this heading, $\Gamma_1(r, \theta, z; r', \theta', z')$, is a solution of the inhomogeneous equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\Gamma_1 = -\frac{\delta(r-r')}{r}\delta(\theta-\theta')\,\delta(z-z'), \tag{15}$$

in the domain $0 < r, r' < \infty$, $0 < \theta, \theta' < \pi$, $-\infty < z, z' < 0$, with a first-order singularity at the point (r', θ', z'); it has a vanishing normal (or θ) derivative at both $\theta = 0$ and $\theta = \pi$, while

$$\partial\Gamma_1/\partial z = K\Gamma_1 \quad (z = 0), \tag{16}$$

and

$$\Gamma_1 \to 0 \quad (z \to -\infty). \tag{17}$$

If the condition $\partial_\theta \Gamma_1 = 0$, $\theta = 0, \pi$, is replaced by one of simple periodicity in θ over the full angular range $0 < \theta < 2\pi$, the corresponding function Γ_0 represents a source in a fluid of infinite lateral extension and depth without any immersed surfaces, and moreover

$$\Gamma_1(r, \theta, z; r', \theta', z') = \Gamma_0(r, \theta, z; r', \theta', z') + \Gamma_0(r, \theta, z; r', -\theta', z'). \tag{18}$$

The determination of Γ_1 can thus be made to devolve upon that of Γ_0, and in the latter connexion it is convenient to start with a Fourier expansion

$$\Gamma_0(r, \theta, z; r', \theta', z') = \sum_{n=-\infty}^{\infty} F_n(r, r', z, z')\, e^{in(\theta-\theta')}, \tag{19}$$

where

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)F_n = -\frac{1}{2\pi}\frac{\delta(r-r')}{r}\delta(z-z').$$

Multiplication in the equation for F_n by $rJ_n(kr)$, where J_n denotes the Bessel function of order n, and subsequent rearrangement (based on integration by parts and use of the Bessel differential equation) establishes that

$$(\partial^2/\partial z^2 - k^2)\,\bar{F}_n = -(1/2\pi)\,J_n(kr')\,\delta(z-z'), \tag{20}$$

where
$$\bar{F}_n = \int_0^\infty rJ_n(kr)\,F_n(r,r',z,z')\,dr$$

is the Fourier–Bessel transform of F_n. The solution of (20), in consort with the requirements (16), (17) turns out to be

$$\bar{F}_n = \frac{1}{4\pi k}J_n(kr')\left[e^{-k|z-z'|} + \frac{k+K}{k-K}e^{k(z+z')}\right], \tag{21}$$

and, on inversion,

$$\begin{aligned} F_n &= \int_0^\infty kJ_n(kr)\,\bar{F}_n\,dk \\ &= \frac{1}{4\pi}\int_0^\infty J_n(kr)\,J_n(kr')\left[e^{-k|z-z'|} + \frac{k+K}{k-K}e^{k(z+z')}\right]dk. \end{aligned} \tag{22}$$

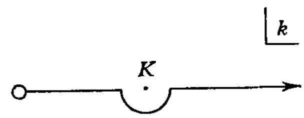

FIGURE 2. The integration path in the k-plane.

With the help of the addition formula,

$$J_0(k\{r^2+r'^2-2rr'\cos(\theta-\theta')\}^{\frac{1}{2}}) \equiv J_0(kD_-) = \sum_{n=-\infty}^{\infty} J_n(kr)\,J_n(kr')\,e^{in(\theta-\theta')},$$

it follows that

$$\begin{aligned} \Gamma_0 &= \frac{1}{4\pi}\int_0^\infty J_0(kD_-)\left[e^{-k|z-z'|} + \frac{k+K}{k-K}e^{k(z+z')}\right]dk \\ &= \frac{1}{4\pi}\left\{\frac{1}{R_{--}} + \frac{1}{R_{-+}} + 2K\int_0^\infty \frac{J_0(kD_-)}{k-K}e^{k(z+z')}\,dk\right\}, \end{aligned} \tag{23}$$

where the first and the second of the subscripts affixed to R distinguish between the combinations $\theta \mp \theta'$, $z \mp z'$, respectively, viz.

$$\left.\begin{aligned} R^2_{--} &= r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2 = D^2_- + (z-z')^2, \\ R^2_{-+} &= r^2+r'^2-2rr'\cos(\theta-\theta')+(z+z')^2 = D^2_- + (z+z')^2. \end{aligned}\right\} \tag{24}$$

The integral in (23) has a pole at $k = K$, which is to be by-passed from below in the complex k-plane, as shown in figure 2, in keeping with the outgoing wave (or radiation) condition for the surface-wave component arising therefrom. Writing

$$I = \int_0^\infty \frac{J_0(kD_-)}{k-K}e^{k(z+z')}\,dk = \frac{1}{2}\int_0^\infty \frac{H_0^{(1)}(kD_-)+H_0^{(2)}(kD_-)}{k-K}e^{k(z+z')}\,dk = \tfrac{1}{2}(I_1+I_2), \tag{25}$$

where $H_0^{(1)}$, $H_0^{(2)}$ designate zero-order Hankel functions of the first and second kinds, the integrals I_1, I_2 can be recast into forms reflecting contours along the positive or negative imaginary axes of the k-plane, respectively; to the former there must be appended a contribution from the residue at the pole $k = K$. Explicitly,

$$I_1 = \int_0^\infty \frac{H_0^{(1)}(kD_-)}{k-K} e^{k(z+z')} dk = 2\pi i H_0^{(1)}(KD_-) e^{K(z+z')} + i\int_0^\infty \frac{H_0^{(1)}(e^{\frac{1}{2}i\pi}\eta D_-)}{i\eta - K} e^{i\eta(z+z')} d\eta,$$

and

$$I_2 = \int_0^\infty \frac{H_0^{(2)}(kD_-)}{k-K} e^{k(z+z')} dk = i\int_0^\infty \frac{H_0^{(2)}(e^{-\frac{1}{2}i\pi}\eta D_-)}{i\eta + K} e^{-i\eta(z+z')} d\eta$$

$$= -i\int_0^\infty \frac{H_0^{(1)}(e^{\frac{1}{2}i\pi}\eta D_-)}{i\eta + K} e^{-i\eta(z+z')} d\eta,$$

whence

$$I = \pi i H_0^{(1)}(KD_-) e^{K(z+z')} - \frac{2}{\pi}\int_0^\infty K_0(\eta D_-) \frac{K\cos\eta(z+z') - \eta\sin\eta(z+z')}{\eta^2 + K^2} d\eta, \tag{26}$$

as

$$K_0(x) = \tfrac{1}{2} i\pi H_0^{(1)}(ix).$$

Collecting the results embodied in (23), (25), (26) and returning to (18), the even source-function Γ_1 is found to be

$$\Gamma_1(r,\theta,z;\,r',\theta',z')$$
$$= \frac{1}{4\pi}\left(\frac{1}{R_{--}} + \frac{1}{R_{+-}} + \frac{1}{R_{-+}} + \frac{1}{R_{++}}\right) + \tfrac{1}{2} iK\{H_0^{(1)}(KD_-) + H_0^{(1)}(KD_+)\} e^{K(z+z')}$$
$$- \frac{K}{\pi^2}\int_0^\infty [K_0(\eta D_-) + K_0(\eta D_+)] \frac{K\cos\eta(z+z') - \eta\sin\eta(z+z')}{\eta^2 + K^2} d\eta, \tag{27}$$

where, in keeping with the notation of (24),

$$\left.\begin{aligned} R_{+-}^2 &= r^2 + r'^2 - 2rr'\cos(\theta+\theta') + (z-z')^2 = D_+^2 + (z-z')^2 \\ R_{++}^2 &= D_+^2 + (z+z')^2. \end{aligned}\right\} \tag{28}$$

and

3. Source-function of odd symmetry

The next objective is to determine a function $\Gamma_2(r,\theta,z;\,r',\theta',z')$ which satisfies the differential equation (15) and the conditions (16), (17) applicable to Γ_1, but with the mixed boundary values

$$\partial_\theta \Gamma_2 = 0, \quad \theta = 0; \quad \Gamma_2 = 0, \quad \theta = \pi, \tag{29}$$

instead of the uniform one assumed by the derivative of Γ_1. In view of the requirements (29), it is appropriate to employ the expansion

$$\Gamma_2 = \sum_{n=0}^{\infty} G_n(r,r',z,z')\cos(n+\tfrac{1}{2})\theta\cos(n+\tfrac{1}{2})\theta', \tag{30}$$

whose individual trigonometric factors, $\cos(n+\frac{1}{2})\theta$, $n = 0, 1, 2, \ldots$, are in accord with (29), and to assure the differential character of Γ_2 through the subsidiary relation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(n+\frac{1}{2})^2}{r^2} + \frac{\partial^2}{\partial z^2}\right) G_n = -\frac{2}{\pi}\frac{\delta(r-r')}{r}\delta(z-z'). \tag{31}$$

From (31) it follows that

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)\bar{G}_n = -\frac{2}{\pi}J_{n+\frac{1}{2}}(kr')\,\delta(z-z') \tag{32}$$

and [compare (20), (21)]

$$\bar{G}_n = \int_0^\infty rJ_{n+\frac{1}{2}}(kr)\,G_n\,dr = \frac{1}{\pi k}J_{n+\frac{1}{2}}(kr')\left[e^{-k|z-z'|}+\frac{k+K}{k-K}e^{k(z+z')}\right], \tag{33}$$

whence

$$G_n = \int_0^\infty kJ_{n+\frac{1}{2}}(kr)\,\bar{G}_n\,dk = \frac{1}{\pi}\int_0^\infty J_{n+\frac{1}{2}}(kr)\,J_{n+\frac{1}{2}}(kr')\left[e^{-k|z-z'|}+\frac{k+K}{k-K}e^{k(z+z')}\right]dk. \tag{34}$$

After substitution of this explicit form of G_n in (30), it appears that

$$\Gamma_2(r,\theta,z;\,r',\theta',z') = P(r,r',z,z';\,\theta-\theta')+P(r,r',z,z';\,\theta+\theta'), \tag{35}$$

with

$$P(r,r',z,z';\,\theta) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^\infty dkJ_{n+\frac{1}{2}}(kr)\,J_{n+\frac{1}{2}}(kr')\cos\left(n+\tfrac{1}{2}\right)\theta\left[e^{-k|z-z'|}+\frac{k+K}{k-K}e^{K(z+z')}\right]. \tag{36}$$

To effect the summation encountered herein, the development

$$\frac{\sin k(r^2+r'^2-2rr'\cos\psi)^{\frac{1}{2}}}{(r^2+r'^2-2rr'\cos\psi)^{\frac{1}{2}}} \equiv \frac{\sin kD(\psi)}{D(\psi)} = \frac{\pi}{2(rr')^{\frac{1}{2}}}\sum_{n=0}^{\infty}(2n+1)\,J_{n+\frac{1}{2}}(kr)\,J_{n+\frac{1}{2}}(kr')\,P_n(\cos\psi)$$

is called upon for establishing the integral representation

$$\frac{\pi}{(rr')^{\frac{1}{2}}}J_{n+\frac{1}{2}}(kr)\,J_{n+\frac{1}{2}}(kr') = \int_0^\pi \frac{\sin kD(\psi)}{D(\psi)}\sin\psi P_n(\cos\psi)\,d\psi, \tag{37}$$

and with the help of the result

$$\sum_{n=0}^{\infty}\cos\left(n+\tfrac{1}{2}\right)\theta P_n(\cos\psi) = \begin{cases}[2(\cos\theta-\cos\psi)]^{-\frac{1}{2}}, & 0\leqslant\theta<\psi<\pi,\\ 0, & 0<\psi<\theta<\pi,\end{cases} \tag{38}$$

there obtains

$$P(r,r',z,z';\,\theta) = \frac{(rr')^{\frac{1}{2}}}{2\pi^2}\int_0^\infty dk\int_\theta^\pi \frac{\sin\psi\,d\psi}{\{2(\cos\theta-\cos\psi)\}^{\frac{1}{2}}}\,\frac{\sin kD(\psi)}{D(\psi)}\left[e^{-k|z-z'|}+\frac{k+K}{k-K}e^{k(z+z')}\right]$$

or

$$P(r,r',z,z';\,\theta) = \frac{1}{2\pi^2}\int_0^\infty dk\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}\theta} dv\,\frac{\sin k(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}}{(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}}\left[e^{-k|z-z'|}+\frac{k+K}{k-K}e^{k(z+z')}\right] \quad (0<\theta<\pi), \tag{39}$$

after the change of variable $\cos\psi = \cos\theta - v^2/2rr'$; the latter form is also valid when $\pi<\theta<2\pi$.

On taking cognizance of the relations

$$\int_0^\infty \frac{\sin kD}{D} e^{-k|z-z'|}\,dk = \frac{1}{D^2+(z-z')^2},$$

and

$$\int_0^\infty \frac{\sin kD}{D}\frac{e^{k(z+z')}}{k-K}\,dk$$
$$= \frac{\pi}{D} e^{iKD} e^{K(z+z')} - \frac{1}{D}\int_0^\infty e^{-\eta D}\frac{K\cos\eta(z+z')-\eta\sin\eta(z+z')}{\eta^2+K^2}\,d\eta \quad (z+z' < 0)$$

(the latter obtained along lines with parallels in the derivation of (26)), a simpler version of P can be given, namely

$$P(r,r',z,z';\theta)$$
$$= \frac{1}{2\pi^2}\left[\frac{\tan^{-1}\{\cos\frac{1}{2}\theta/(\sigma_-^2-\cos^2\frac{1}{2}\theta)^{\frac{1}{2}}\}}{\{r^2+r'^2-2rr'\cos\theta+(z-z')^2\}^{\frac{1}{2}}}+\frac{\tan^{-1}\{\cos\frac{1}{2}\theta/(\sigma_+^2-\cos^2\frac{1}{2}\theta)^{\frac{1}{2}}\}}{\{r^2+r'^2-2rr'\cos\theta+(z+z')^2\}^{\frac{1}{2}}}\right]$$
$$+\frac{K}{\pi}e^{K(z+z')}\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}\theta}\frac{\exp[iK(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}]}{(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}}\,dv$$
$$-\frac{K}{\pi^2}\int_0^\infty d\eta\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}\theta}dv\,\frac{\exp[-\eta(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}]}{(r^2+r'^2-2rr'\cos\theta+v^2)^{\frac{1}{2}}}$$
$$\times\frac{K\cos\eta(z+z')-\eta\sin\eta(z+z')}{\eta^2+K^2}, \tag{40}$$

wherein

$$\sigma_\mp^2 = \{(r+r')^2+(z\mp z')^2\}/4rr'. \tag{41}$$

Finally, in consequence of (35),

$$\Gamma_2(r,\theta,z;r',\theta',z')$$
$$= \frac{1}{2\pi^2}\left[\frac{1}{R_{--}}\tan^{-1}\frac{\cos\frac{1}{2}(\theta-\theta')}{\{\sigma_-^2-\cos^2\frac{1}{2}(\theta-\theta')\}^{\frac{1}{2}}}+\frac{1}{R_{-+}}\tan^{-1}\frac{\cos\frac{1}{2}(\theta-\theta')}{\{\sigma_+^2-\cos^2\frac{1}{2}(\theta-\theta')\}}\right.$$
$$\left.+\frac{1}{R_{+-}}\tan^{-1}\frac{\cos\frac{1}{2}(\theta+\theta')}{\{\sigma_-^2-\cos^2\frac{1}{2}(\theta+\theta')\}^{\frac{1}{2}}}+\frac{1}{R_{++}}\tan^{-1}\frac{\cos\frac{1}{2}(\theta+\theta')}{\{\sigma_+^2-\cos^2\frac{1}{2}(\theta+\theta')\}^{\frac{1}{2}}}\right]$$
$$+\frac{K}{\pi}e^{K(z+z')}\left\{\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta')}\frac{\exp[iK(D_-^2+v^2)^{\frac{1}{2}}]}{(D_-^2+v^2)^{\frac{1}{2}}}\,dv\right.$$
$$\left.+\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta')}\frac{\exp[iK(D_+^2+v^2)^{\frac{1}{2}}]}{(D_+^2+v^2)^{\frac{1}{2}}}\,dv\right\}$$
$$-\frac{K}{\pi^2}\int_0^\infty d\eta\,\frac{K\cos\eta(z+z')-\eta\sin\eta(z+z')}{\eta^2+K^2}$$
$$\times\left\{\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta')}\frac{\exp[-\eta(D_-^2+v^2)^{\frac{1}{2}}]}{(D_-^2+v^2)^{\frac{1}{2}}}\,dv\right.$$
$$\left.+\int_0^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta')}\frac{\exp[-\eta(D_+^2+v^2)^{\frac{1}{2}}]}{(D_+^2+v^2)^{\frac{1}{2}}}\,dv\right\}. \tag{42}$$

4. Complete source function

The superposition—or, more precisely, the average—of Γ_1 and Γ_2 defines a regular potential function everywhere in the fluid, except for an isolated (first-order) singularity at the point (r', θ', z'), and is such that the requirements of constant pressure at the free surface ($z = 0$), vanishing normal velocity at the rigid half-plane ($\theta = 0, 2\pi$) and quiescence at great depths ($z \to -\infty$) are all met; it therefore represents the complete source function for the configuration contemplated. Explicitly,

$$
\begin{aligned}
G(r,\theta,z;r',\theta',z') &= \frac{\Gamma_1+\Gamma_2}{2} \\
&= G_N(r,r',\theta,\theta',z-z') + G_N(r,r',\theta,\theta',z+z') \\
&\quad + \frac{K}{2\pi} e^{K(z+z')} \left\{ \int_{-\infty}^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta')} \frac{\exp[iK(D_-^2+v^2)^{\frac{1}{2}}]}{(D_-^2+v^2)^{\frac{1}{2}}} dv \right. \\
&\quad \left. + \int_{-\infty}^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta')} \frac{\exp[iK(D_+^2+v^2)^{\frac{1}{2}}]}{(D_+^2+v^2)^{\frac{1}{2}}} dv \right\} \\
&\quad - \frac{K}{2\pi^2} \int_0^\infty d\eta \frac{K\cos\eta(z+z') - \eta\sin\eta(z+z')}{\eta^2+K^2} \\
&\quad \times \left\{ \int_{-\infty}^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta')} \frac{\exp[-\eta(D_-^2+v^2)^{\frac{1}{2}}]}{(D_-^2+v^2)^{\frac{1}{2}}} dv \right. \\
&\quad \left. + \int_{-\infty}^{2(rr')^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta')} \frac{\exp[-\eta(D_+^2+v^2)^{\frac{1}{2}}]}{(D_+^2+v^2)^{\frac{1}{2}}} dv \right\}, \qquad (43)
\end{aligned}
$$

where G_N is the three-dimensional harmonic Neumann Green's function (or fundamental solution of the second boundary-value problem) for the whole of space, with a semi-infinite plane embedded therein, viz.

$$
\begin{aligned}
G_N(r,r',\theta,\theta',z-z') &= \frac{1}{8\pi} \left\{ \frac{1}{R_{--}} \left(1 + \frac{2}{\pi} \tan^{-1} \frac{\cos\frac{1}{2}(\theta-\theta')}{\{\sigma_-^2 - \cos^2\frac{1}{2}(\theta-\theta')\}^{\frac{1}{2}}} \right) \right. \\
&\quad \left. + \frac{1}{R_{+-}} \left(1 + \frac{2}{\pi} \tan^{-1} \frac{\cos\frac{1}{2}(\theta+\theta')}{\{\sigma_-^2 - \cos^2\frac{1}{2}(\theta+\theta')\}^{\frac{1}{2}}} \right) \right\} \\
&\qquad (-\infty < z-z' < \infty), \qquad (44) \\
\partial_\theta G_N &= 0, \quad \theta = 0, 2\pi;
\end{aligned}
$$

in arriving at the particular form of G given above, use has been made of the integral representation

$$H_0^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\exp[i(z^2+v^2)^{\frac{1}{2}}]}{(z^2+v^2)^{\frac{1}{2}}} dv \quad (z > 0)$$

and of a similar representation for $K_0(z)$.

The source function (43) differs from that obtained by Voit in some important particulars: first, in respect of its validity at all points of the fluid (whereas Voit's representation (7) applies only to the stratum between the free surface and the depth level of the source), and secondly, in that the surface-wave component is compactly rendered by means of the two single integrals,

$$G_{\text{surface wave}} = \frac{K}{2\pi} e^{K(z+z')} \left\{ \int_{-\infty}^{2(rr')^{\frac{1}{2}} \cos \frac{1}{2}(\theta-\theta')} \frac{\exp[iK(D_-^2+v^2)^{\frac{1}{2}}]}{(D_-^2+v^2)^{\frac{1}{2}}} dv + \int_{-\infty}^{2(rr')^{\frac{1}{2}} \cos \frac{1}{2}(\theta+\theta')} \frac{\exp[iK(D_+^2+v^2)^{\frac{1}{2}}]}{(D_+^2+v^2)^{\frac{1}{2}}} dv \right\}, \quad (45)$$

inasmuch as the functions G_N are of a static (or non-propagating) nature, while the double integrals make up a wave-free component that diminishes more rapidly than (45) with increasing lateral distance from the source.† In (45), the (real) exponential factor depending on the depths z, z' of the observation and source points is characteristic for a surface excitation, and the terms in braces constitute the integral representation of a two-dimensional Green's function $g(r,\theta;r',\theta')$, such that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + K^2\right) g(r,\theta;r',\theta') = -4\pi \frac{\delta(r-r')}{r} \delta(\theta-\theta') \quad (0 < \theta < 2\pi),$$

and

$$\partial_\theta g = 0, \quad \theta = 0, 2\pi,$$

this being the acoustic velocity potential for a line source at (r',θ') in the presence of a rigid half-plane. The line source function (whose singularity is a logarithmic one) does not feature a coefficient K as appears in (45), and it will be perceived there that an increase in the magnitude of K $(=\omega^2/g)$ is, however, offset by a corresponding reduction in the range of appreciable surface wave amplitude. An approximation to (45) for large values of $K(rr')^{\frac{1}{2}} \cos \frac{1}{2}(\theta \mp \theta')$ in the domains $0 < \theta < \pi - \theta'$, $\pi - \theta' < \theta < \pi + \theta'$, $\pi + \theta' < \theta < 2\pi$ is easily come by, and apart from direct or image line-source terms in the first two domains, the edge-scattered wave potential to lowest order leads to a surface displacement of the form (14).

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† The integrals with respect to η over an infinite range can be reduced to finite and more convenient forms by noting that if

$$F(\alpha) = \int_0^\infty \frac{K\cos\eta\alpha - \eta\sin\eta\alpha}{\eta^2+K^2} e^{-\eta\beta} d\eta \quad (\alpha < 0, \beta > 0),$$

then

$$\frac{dF}{d\alpha} - KF = -\int_0^\infty \cos\eta\alpha \, e^{-\eta\beta} d\eta = -\frac{\beta}{\alpha^2+\beta^2},$$

whence

$$F(\alpha) = F(0) + \beta e^{K\alpha} \int_0^{-\alpha} \frac{e^{K\zeta}}{\zeta^2+\beta^2} d\zeta,$$

with

$$F(0) = [\sin K\beta \,\mathrm{Ci}(K\beta) - \cos K\beta\{\mathrm{Si}(K\beta) - \tfrac{1}{2}\pi\}].$$